$y = x$ is a particular solution of $3x^2y'' + 8xy' - 2y = 6x$. **SCORE: _____ / 10 PTS**

[a] By inspection, find a particular solution of $3x^2y'' + 8xy' - 2y = 1$.

$-2y = 1 \longrightarrow y = -\frac{1}{2}$ $\frac{1}{2}$

[b] **Using superposition and linearity**, find the general solution of $3x^2y'' + 8xy' - 2y = -12x + 3$.

$-2(6x)$

$3(1)$

① $3r^2 + 5r - 2 = 0$

$(3r-1)(r+2) = 0$

$r = \frac{1}{3}, -2$ ①

$y = -2(x) + 3(-\frac{1}{2}) + Ax^{\frac{1}{3}} + Bx^{-2}$

$= -2x - \frac{3}{2} + Ax^{\frac{1}{3}} + Bx^{-2}$ ① ① ①

[c] Solve the initial value problem, $3x^2y'' + 8xy' - 2y = -12x + 3, \quad y(-1) = 1, \quad y'(-1) = -2$.

$y' = -2 + \frac{1}{3}Ax^{-\frac{2}{3}} - 2Bx^{-3}$ ①

$2 - \frac{3}{2} - A + B = 1$

$-2 + \frac{1}{3}A + 2B = -2$

$-A + B = \frac{1}{2}$

$\frac{1}{3}A + 2B = 0$

$A = -6B$

$7B = \frac{1}{2}$

$B = \frac{1}{14}$

$A = -\frac{3}{7}$

$y = -2x - \frac{3}{2} - \frac{3}{7}x^{\frac{1}{3}} + \frac{1}{14}x^{-2}$ $\frac{1}{2}$ ① ①

[d] According to the existence and uniqueness theorem,
what is the largest interval over which the initial value problem in [c] is guaranteed to have a unique solution ?

$y'' + \frac{8}{3x}y' - \frac{2}{3x^2}y = -\frac{4}{x} + \frac{1}{x^2}$

COEF'S DISCONT @ $x = 0$

① $(-\infty, 0)$

Find the general solution of $3y''' + 8y'' + 12y' - 5y = 0$. **SCORE: _____ / 6 PTS**

$$3r^3 + 8r^2 + 12r - 5 = 0 \quad \frac{1}{2}$$

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & 8 & 12 & -5 \\ & & 1 & 3 & 5 \\ \hline & 3 & 9 & 15 & 0 \end{array}$$

$$1 \quad \left(r - \frac{1}{3}\right)(3r^2 + 9r + 15) = 0 \quad 1$$

$$(3r - 1)(r^2 + 3r + 5) = 0$$

$$r = \frac{-3 \pm \sqrt{-11}}{2} = -\frac{3}{2} \pm \frac{\sqrt{11}}{2} i \quad \frac{1}{2}$$

$$y = \overset{1}{Ae^{\frac{1}{3}x}} + Be^{-\frac{3}{2}x} \cos \frac{\sqrt{11}}{2} x \quad 1$$

$$+ Ce^{-\frac{3}{2}x} \sin \frac{\sqrt{11}}{2} x \quad 1$$

**Use the Wronskian** to determine if the functions $f_1(x) = x^2 - 2x$, $f_2(x) = 1 - 2x^2$ and $f_3(x) = 1 - 4x$ are linearly independent on $(-\infty, \infty)$. **State your justification and conclusion BRIEFLY & clearly.**

SCORE: _____ / 4 PTS

$$\begin{vmatrix} x^2-2x & 1-2x^2 & 1-4x \\ 2x-2 & -4x & -4 \\ 2 & -4 & 0 \end{vmatrix} = 0 + (-8)(1-2x^2) - 4(1-4x)(2x-2) - (-8x)(1-4x) - 16(x^2-2x) - 0$$

$$= \underbrace{-8+16x^2}_{\frac{1}{2}} \underbrace{+8-40x+32x^2}_{1} \underbrace{+8x-32x^2}_{\frac{1}{2}} \underbrace{-16x^2+32x}_{\frac{1}{2}}$$

$$\frac{1}{2} \; \underbrace{= 0}$$

1 NOT LINEARLY INDEPENDENT.

If $x = e^t$, write and prove a formula for $\frac{d^3y}{dx^3}$ in terms of $x$, $\frac{dy}{dt}$, $\frac{d^2y}{dt^2}$ and/or $\frac{d^3y}{dt^3}$. **SCORE: _____ / 4 PTS**

Your proof may use the formula for $\frac{d^2y}{dx^2}$ from lecture without proving it.

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left[-\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}\right]$$

$$= \frac{2}{x^3}\frac{dy}{dt} \underbrace{- \frac{1}{x^2}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}}_{1} - \frac{2}{x^3}\frac{d^2y}{dt^2} + \underbrace{\frac{1}{x^2}\frac{d}{dt}\left(\frac{d^2y}{dt^2}\right)\frac{dt}{dx}}_{1}$$

$$= \frac{2}{x^3}\frac{dy}{dt} - \frac{1}{x^2}\frac{d^2y}{dt^2}\frac{1}{x} - \frac{2}{x^3}\frac{d^2y}{dt^2} + \frac{1}{x^2}\frac{d^3y}{dt^3}\frac{1}{x}$$

$$= \underbrace{\frac{2}{x^3}\frac{dy}{dt}}_{\frac{1}{2}} \underbrace{- \frac{3}{x^3}\frac{d^2y}{dt^2}}_{1} + \underbrace{\frac{1}{x^3}\frac{d^3y}{dt^3}}_{\frac{1}{2}}$$

$y_1 = e^{-x}$ is a solution of $xy'' + (2x-1)y' + (x-1)y = 0$. Find a second linearly independent solution. **SCORE: _____ / 6 PTS**

$y_2 = ve^{-x}$

$y_2' = v'e^{-x} - ve^{-x}$

$y_2'' = v''e^{-x} - v'e^{-x} - v'e^{-x} + ve^{-x}$

$= v''e^{-x} - 2v'e^{-x} + ve^{-x}$

$\underbrace{x(v''e^{-x} - 2v'e^{-x} + ve^{-x})}$, 1

$\underbrace{+(2x-1)(v'e^{-x} - ve^{-x})}$, 1

$\underbrace{+(x-1)(ve^{-x})}$, $\frac{1}{2}$

$= e^{-x}(xv'' - 2xv' + xv + (2x-1)v' - (2x-1)v + (x-1)v)$

$= \underbrace{e^{-x}(xv'' - v')} = 0$ 1

LET $u = v'$

$xu' - u = 0$

$xu' = u$

$\underbrace{\frac{1}{u}\,du = \frac{1}{x}\,dx}$ 1

$\ln|u| = \ln|x|$

$v' = \underbrace{u = x}$ $\frac{1}{2}$

$\underbrace{v = \frac{1}{2}x^2}$ $\frac{1}{2}$

$y_2 = \underbrace{\frac{1}{2}x^2e^{-x}}_{\frac{1}{2}}$ OR $\underbrace{x^2e^{-x}}_{\frac{1}{2}}$

**BONUS QUESTION:** Find all values of $\ln(-1)$.

**SCORE:** _____ **/ 3 PTS**

$$e^{ix} = \cos x + i \sin x = -1 \longrightarrow \begin{matrix} \cos x = -1 \\ \sin x = 0 \end{matrix} \longrightarrow x = \pi + 2n\pi$$

$$\ln(-1) = \underbrace{i\pi}_{1\frac{1}{2}} + \underbrace{2ni\pi}_{1\frac{1}{2}}$$